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Buoyant convection from horizontal surfaces at constant pressure

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A two-dimensional horizontal flow is discussed, which is induced by other, buoyancy-driven flows elsewhere. It is an adaptation of the incompressible wall jet, which is driven by conditions at the leading edge and has no streamwise pressure gradient. The relation of this flow to the classical buoyancy-driven boundary layers on inclined and horizontal surfaces is investigated, as well as its possible connexion with a two-dimensional buoyant plume driven by a line source of heat. Composite flows are constructed by patching various such solutions together. The composite flows exhibit $Gr^{\frac{1}{4}}$ scaling (Gr being the Grashof number).

1. Introduction

Flows driven by buoyancy are not necessarily directed predominately upwards, even when they are unconfined in the vertical direction by a restraining boundary. Devotees of idly staring at log fires are familiar with the curling of a flame around a log, often into a nearly horizontal direction.

Stewartson (1958) gave a theoretical description of a flow field induced by buoyancy on a heated horizontal semi-infinite flat plate. The flow is self-similar, and corresponds to horizontal, accelerating boundary-layer flow below the plate, directed away from the leading edge. Since the buoyant driving force has in this case no component along the surface, the accelerating flow must be driven indirectly by a buoyancy-induced pressure gradient. Stewartson's analysis contains a sign error in the pressure-gradient term of the horizontal momentum equation. When this is corrected, one finds a horizontal boundary layer above the heated surface, driven by an induced pressure, which falls in the direction of flow. The correct analysis was given by Gill, Zeh & del-Casal (1965). The same boundary-layer flow cannot exist below the heated plate, for it is associated with rising pressure in the flow direction. Extensive numerical integrations of the governing equations for the allowed flow were carried out by Rotem & Claassen (1969), who gave detailed velocity and temperature profiles as a function of Prandtl number. Their study also included results of a flow-visualization experiment. This shows that, although a horizontal boundary layer clearly can exist above a horizontal surface, its horizontal extent is limited.

The flow just described differs in several ways from that driven by buoyancy on a semi-infinite flat plate inclined to the horizontal. Here the buoyancy force has a component along the surface, which directly drives the flow. It can be shown that, except in the neighbourhood of the leading edge, this direct drive dominates induced pressure effects, which can therefore be neglected (Jones 1973). The resulting flow essentially reduces to the classical case of free convection on a vertical surface, which was discussed in detail by Ostrach (1964). Again a similarity solution exists, though with different form of the similarity variables from the horizontal case. In particular, for the horizontal boundary layer, the velocity along the surface varies as $g^{\frac{2}{5}}\nu^{\frac{1}{5}}$, where g is the acceleration of gravity and ν is kinematic viscosity. For the inclined case, the streamwise velocity varies as $g^{\frac{1}{2}}$ and is independent of viscosity. This is because the boundary-layer variables are stretched with $Gr^{\frac{1}{5}}$ for the horizontal case, and with $Gr^{\frac{1}{4}}$ for the inclined case (Gr = Grashof number). For convenience, this horizontal flow will be called an SRC layer and the inclined flow an O layer. The O-layer streamwise velocity increases more rapidly along the surface than that of the SRC layer.

A combination of these is discussed in Jones (1973). In considering a very slightly inclined semi-infinite surface, it was noticed that the region near the leading edge resembles an SRC layer, while far downstream the flow asymptotically corresponds to the O layer. This results from the fact that omission of the direct driving term is disallowed only far downstream when an SRC-layer scaling is used. Likewise, the neglect of induced pressure only becomes invalid far upstream with an O-layer scaling. Jones (1973) gives non-similar corrections which bridge the gap between the two different similarity solutions to produce a smooth transition between them. Pera & Gebhart (1973*a*) use the same technique to correct the upstream solution; and their study contains a comparison of their calculations with experimental velocity and temperature profiles.

The purpose of this paper is to point out the possibility of another horizontal flow on a heated horizontal surface, one in which the velocity is uninfluenced by any pressure gradient. In this case there is neither a direct nor indirect drive from buoyancy; and this raises a very basic question about the existence of any flow at all. This question is easily answered: it was shown by Glauert (1956, 1958), for an isothermal flow, that there is indeed a non-vanishing solution of the boundary-layer equations, with uniform pressure and vanishing streamwise velocity both at the surface and far from it. This kind of flow is known as a *wall jet*; and it gets its drive from conditions imposed at the leading edge. Its streamwise velocity decreases with increasing downstream distance. Qualitatively, the wall-jet velocity profile in the transverse direction resembles those of the buoyancy-driven flows just described. A deeper correspondence with buoyancy-driven flow in fact exists. In § 2 it is shown that, if the O-layer scaling is retained for the heated nearly horizontal surface, then the streamwise momentum equation can be transformed into Glauert's equation for the incompressible wall jet. The appropriate similarity form also allows a consistent balancing in the transverse direction of pressure gradient and buoyancy, and an energy balance, from which the momentum equation is decoupled. This flow will be called a G layer. Its properties are described in § 3. The velocity field is available analytically from the solution of Glauert (1956, 1958). The temperature field also can be inferred from solutions already obtained by Riley (1958) for the compressible wall jet.

The G layer is mathematically identical to Glauert's wall jet, but it appears here in a different context, since it owes its existence indirectly to gravitational body forces. As a wall jet, it needs to be driven by conditions at the leading edge. One drive would be a buoyancy-induced flow on a heated inclined upstream portion of the surface, i.e. an upstream O layer. A calculation which patches an O layer to a G layer is given in §4. A smooth matching with a non-similar transition is beyond the scope of the paper; but it is interesting that, for both flows, the streamwise velocity is found to vary as $g^{\frac{1}{2}}$, and to be independent of viscosity. Since both scale with $Gr^{\frac{1}{2}}$, the composite flow contains contributions from each whose relative weight is independent of Gr and ν. This is in contrast to Jones' (1973) matching of an SRC layer to an O layer.

Another interesting property of the G layer is found in connexion with two-dimensional vertical buoyant plumes. It is known experimentally that horizontal flow on a horizontal heated semi-infinite surface remains attached only over a limited length (Rotem & Claassen 1969; Pera & Gebhart 1973*b*). In most situations, in fact, vertical plumes are the most obvious element in buoyant convection above horizontal or nearly horizontal surfaces. Horizontal flow along the surface must play an essential role in feeding such plumes. A particularly clear illustration of this is displayed in the flow-visualization experiments reported by Husar & Sparrow (1968). In §6 it is shown that a compatible scaling exists between the G layer and the similarity form of the classical two-dimensional vertical plume driven by a line source of heat (Brand & Lahey 1967; Gebhart, Pera & Schorr 1970; Fujii, Morioka & Uehara 1973), if one equates the heat transferred to the G layer over a given horizontal length to the strength of the heat source driving the plume. These flows can then also be patched together, to give a vertical plume velocity which varies as $g^{\frac{1}{2}}$, and becomes independent of viscosity. Here, too, no real matching calculation is attempted; but it is again clear that, for any such composite flow, the relative contributions of G layer and plume would be independent of Gr.

2. Governing equation and similarity

In the two-dimensional equations of motion, with the Boussinesq approximation, which has become conventional for this problem (Stewartson 1958; Rotem & Claassen 1969), both the temperature gradient in the absence of motion and the adiabatic gradient are ignored:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial \Delta p}{\partial x}+g\sin\alpha\frac{\Delta T}{T_0}+\nu\nabla^2 u, \tag{2}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial \Delta p}{\partial z}+g\cos\alpha\frac{\Delta T}{T_0}+\nu\nabla^2 w, \tag{3}$$

$$u\frac{\partial \Delta T}{\partial x}+w\frac{\partial \Delta T}{\partial z}=K\nabla^2\Delta T. \tag{4}$$

Here g is the gravitational acceleration, while ν and K are the appropriate transport coefficients. Quantities u and w are the components of velocity in the directions x and z, along and normal to a semi-infinite flat plate, as measured from its origin. The plate is inclined to the horizontal at an angle α, and a surface reference temperature is T_1. Far from the plate, the temperature is T_0, and the density is ρ_0. The pressure and temperature fields induced by the motion are

$$\Delta T = T - T_0, \quad \Delta p = p - p_0 + \rho_0 g(x \sin\alpha + z\cos\alpha).$$

The variables are now non-dimensionalized and stretched in the following way, using a characteristic length L along the surface and two dimensionless constants β and ϵ:

$$\hat{x} = \frac{x}{L}, \quad \hat{z} = \frac{z}{L} Gr^{\beta}, \quad \hat{u} = \frac{uL}{\nu} Gr^{-2\beta}, \quad \hat{w} = \frac{wL}{\nu} Gr^{-\beta},$$

$$\Delta\hat{T} = \frac{\Delta T}{T_1 - T_0}, \quad \Delta\hat{p} = \frac{\Delta p L^2}{\rho_0 \nu^2} Gr^{-\epsilon}.$$

Equations (1)–(4) become, in these variables,

$$\frac{\partial\hat{u}}{\partial\hat{x}} + \frac{\partial\hat{w}}{\partial\hat{z}} = 0, \tag{5}$$

$$\hat{u}\frac{\partial\hat{u}}{\partial\hat{x}} + \hat{w}\frac{\partial\hat{u}}{\partial\hat{z}} = -\,Gr^{\epsilon-4\beta}\frac{\partial\Delta\hat{p}}{\partial\hat{x}} + \sin\alpha\, Gr^{1-4\beta}\,\Delta\hat{T} + \frac{\partial^2\hat{u}}{\partial\hat{z}^2} + Gr^{-2\beta}\frac{\partial^2\hat{u}}{\partial\hat{x}^2}, \tag{6}$$

$$\hat{u}\frac{\partial\hat{w}}{\partial\hat{x}} + \hat{w}\frac{\partial\hat{w}}{\partial\hat{z}} = -\,Gr^{\epsilon-2\beta}\frac{\partial\Delta\hat{p}}{\partial\hat{z}} + \cos\alpha\, Gr^{1-3\beta}\,\Delta\hat{T} + \frac{\partial^2\hat{w}}{\partial\hat{z}^2} + Gr^{-2\beta}\frac{\partial^2\hat{w}}{\partial\hat{x}^2}, \tag{7}$$

$$\hat{u}\frac{\partial\Delta\hat{T}}{\partial\hat{x}} + \hat{w}\frac{\partial\Delta\hat{T}}{\partial\hat{z}} = \frac{1}{Pr}\left[\frac{\partial^2\Delta\hat{T}}{\partial\hat{z}^2} + Gr^{-2\beta}\frac{\partial^2\Delta\hat{T}}{\partial\hat{x}^2}\right]. \tag{8}$$

The quantity Gr is the Grashof number, which is supposed to be large, and Pr is the Prandtl number. Gr is given by

$$Gr = \frac{gL^3}{\nu^2}\frac{T_1 - T_0}{T_0}.$$

It can be verified that the scaling used by Ostrach (1964) for the vertical plate corresponds to $\epsilon = 1$, $\beta = \frac{1}{4}$. The scaled pressure gradient across the boundary layer is then, from (7), $O(Gr^{-\frac{1}{4}})$. From this, one needs to argue physically that the longitudinal pressure gradient in (6), though in principle of order unity with this scaling, is in fact that of the outside flow, or nothing. The argument, but not the conclusion, is different for an inclined but not horizontal surface. One can now balance the transverse pressure gradient against the transverse component of the thermal driving term in (7), which gives $\epsilon = 1-\beta$. Since $\beta = \frac{1}{4}$ still, based on the longitudinal thermal driving term in (6), $\epsilon = \frac{3}{4}$. Then, although the transverse gradient of pressure in (7) is not negligible, the horizontal one in (6) is, since this is $O(Gr^{1-5\beta}) = O(Gr^{-\frac{1}{4}})$. In either case, pressure effects in (6) are negligible. For any inclination except horizontal, or nearly so, (6) describes a balance between inertia, shear and direct thermal drive.

For a horizontally oriented surface, the scaling of Stewartson (1958) and Rotem & Claassen (1969) corresponds to $\beta = \frac{1}{5}$ and $\epsilon = \frac{4}{5}$. This again balances transverse thermal drive and transverse pressure gradient in (7), with $\epsilon = 1-\beta$. But now the pressure term in (6) is of order unity, and in fact is needed physically to drive the flow. The balance is between inertia, shear and pressure.

We next introduce similarity variables, excluding from further consideration the vertical or nearly vertical surface. The then permitted balance in (7), between pressure and buoyancy terms, is made into a requirement for such a balance, and thus $\epsilon = 1-\beta$. Further, in discussing the horizontal or nearly horizontal surface it has become conventional to introduce a modified Grashof number $\hat{Gr} = Gr\cos\alpha$. This is also done here. Equations (5)–(8) are then essentially unchanged except as follows: Gr is replaced by $\hat{Gr}$, $\cos\alpha$ disappears from the second term on the right of (7), and $\sin\alpha$ is replaced by $\tan\alpha$ in the second term on the right of (6). The stream function ψ, temperature and pressure are taken to be of the form

$$\psi = \hat{x}^p F(\eta), \quad \Delta\hat{T} = \hat{x}^n H(\eta), \quad \Delta\hat{p} = \hat{x}^{1-p+n}\, G(\eta),$$

where $\eta = \hat{z}\hat{x}^{p-1}$. Equations (6)–(8), modified as described above, then become

$$F''' + pFF'' - (2p-1)\,F'^2 = \hat{Gr}^{1-5\beta}\,\hat{x}^{3+n-5p}\,[(1-p+n)\,G - (1-p)\,\eta G'] \\ - \hat{Gr}^{1-4\beta}\hat{x}^{3+n-4p}\tan\alpha H + O(\hat{Gr}^{-2\beta}\,\hat{x}^{-2p}), \qquad (9)$$

$$G' = H + O(\hat{Gr}^{3\beta-1}\hat{x}^{-(3+n-3p)}), \qquad (10)$$

$$Pr^{-1}H'' + pFH' - nHF' = O(\hat{Gr}^{-2\beta}\hat{x}^{-2p}). \qquad (11)$$

The occurrence of G and H in the first and second terms on the right of (9) identify these as the induced pressure and the direct thermal driving terms. Clearly, a similarity solution for the inclined surface requires $\beta = \frac{1}{4}, p = \frac{1}{4}(3+n)$. This is the O layer. The induced pressure force in (9) then varies as $\hat{Gr}^{-\frac{1}{4}}\,\hat{x}^{-\frac{1}{4}(3+n)}$, i.e. not only is it small for a large $\hat{Gr}$ but it decreases with increasing downstream distance. Equations (9)–(11) then transform into those given of Ostrach (1964). Similarly, for the nearly horizontal surface, if the induced pressure gradient is to drive the flow, one needs $\beta = \frac{1}{5}$, $p = \frac{1}{5}(3+n)$. Equations (9)–(11) are then those of Rotem & Claassen (1969), and the flow is an SRC layer. The direct driving term becomes proportional to $\hat{Gr}^{\frac{1}{5}}\hat{x}^{\frac{1}{5}(3+n)}\tan\alpha$. This sets a limit on the allowable surface inclination for which direct thermal driving is unimportant, and for given α and $\hat{Gr}$ the neglect of this term becomes invalid sufficiently far downstream. This is the behaviour exploited by Jones (1973) in his composite, non-similar solution for the slightly inclined surface.

To motivate the next step physically, consider an example discussed again at the end of § 5. On the upper side of a cooled inclined surface of finite length, colder heavy fluid will flow down the incline, accelerating towards the bottom. If the lower end of the incline terminates on a horizontal plane, the accelerated fluid must flow over this plane under the retarding force of friction but in absence of any further direct thermal drive. Indirectly induced drive by pressure may also exist; but this can be dominated by inertia effects if the magnitude of the velocity attained at the bottom of the incline is sufficiently large. Such a horizontal flow must therefore slow down. One therefore looks for the possibility of a balance between inertia and friction on the horizontal surface, with a scaling compatable with that of the inclined flow, or $\beta = \frac{1}{4}$. With this β all the terms on the right of (9) can be made arbitrarily small for sufficiently small values of α and $\hat{G}r^{-\frac{1}{4}}$. The first-order terms on the left of (9) become identical to those that make up the equation solved by Glauert (1956) to obtain the wall jet if $p = \frac{1}{4}$, $f = \frac{1}{4}F$. This is the only non-trivial solution, assuming no reverse flow. Equations (9)–(11) then become

$$f''' + ff'' + 2f'^2 = -\tfrac{1}{4}\hat{x}^{2+n}\tan\alpha H + O(\hat{G}r^{-\frac{1}{4}}\hat{x}^{\frac{7}{4}+n}), \tag{12}$$

$$G' = H + O(\hat{G}r^{-\frac{1}{4}}\hat{x}^{-(\frac{9}{4}+n)}), \tag{13}$$

$$Pr^{-1}H'' + fH' - 4nf'H = O(\hat{G}r^{-\frac{1}{2}}\hat{x}^{-\frac{1}{2}}). \tag{14}$$

The appropriate boundary conditions are

$$f(0) = f'(0) = f'(\infty) = H(\infty) = G(\infty) = 0, \quad H(0) = \pm 1.$$

The upper and lower signs occurring in the surface boundary condition on temperature correspond to flow above or below a heated surface, respectively, or below and above a cooled surface. In contrast to the case of pressure-driven flow, either sign is possible.

3. Results based on wall-jet analogy

Similarity solutions for (12)–(14) exist in the limit $\hat{G}r \to \infty$, $\alpha \to 0$. Since the boundary conditions as well as the governing equation for f are identical to those of the wall jet, the flow field is given by Glauert (1956, (4.5))

$$\eta = \log\frac{(1+f^{\frac{1}{2}}+f)^{\frac{1}{2}}}{1-f^{\frac{1}{2}}} + \sqrt{3}\tan^{-1}\frac{(3f)^{\frac{1}{2}}}{2+f^{\frac{1}{2}}}, \quad f(\infty) = 1, \quad f''(0) = \tfrac{2}{9}. \tag{15}$$

In the following, the positive sign is taken in the thermal surface boundary condition. Some solutions of (14) were given by Riley (1958). The ones appropriate to the present boundary conditions are

$$n = 0 \quad H = 1 - f^{\frac{1}{2}}, \quad Pr = 1; \tag{16}$$

$$n = -\tfrac{1}{4}, \quad H = (1 - f^{\frac{3}{2}})^{Pr}. \tag{17}$$

The heat transfer to the surface is proportional to $H'(0)\,\hat{x}^{n-\frac{3}{4}}$. Using the fact that $f \simeq \frac{1}{9}\eta^2$ for small η one finds that for (16) $H'(0) = -\frac{1}{3}$. The temperature given by (17) corresponds to $H'(0) = 0$, and therefore an insulated surface with its temperature excess decreasing in the downstream direction as $x^{-\frac{1}{4}}$.

Equation (13) for the pressure is easily integrated for these cases when $Pr = 1$, as elaborated on below. One obtains

$$n = 0, \quad G = -2\sqrt{3}\tan^{-1}\frac{1-f^{\frac{1}{2}}}{\sqrt{3(1+f^{\frac{1}{2}})}}, \tag{16a}$$

$$n = -\tfrac{1}{4}, \quad G = -3(1 - f^{\frac{1}{2}}). \tag{17a}$$

These integrations are facilitated by a transformation to $f^{\frac{1}{2}} = y$ as a new independent variable, following both Glauert and a lead provided by Riley (1958).

This permits a number of other solutions of the energy equation to be obtained easily.† These, while not directly relevant to the present considerations regarding buoyancy-driven flow, are also given below, since they may be of general interest for the compressible wall jet. Because

$$f' = 2yy' = \tfrac{4}{3}y(1-y^3),$$

one has

$$G = -\int_{\eta}^{\infty} H\,d\eta = -3\int_{y}^{1}\frac{H}{1-y^3}\,dy. \tag{18}$$

Once H is known as a function of y, (18) can be integrated directly. The equation for H is, from (14),

$$(1-y^3)\frac{d^2H}{dy^2}+(Pr-1)\,3y^2\frac{dH}{dy}-24n\,Pr\,H = 0. \tag{19}$$

This can be transformed into a hypergeometric equation. For certain values of n, solutions exist that contain only a finite number of terms. For $Pr = 1$, some of these are as follows, in addition to (16) and (17):

$$n = -\tfrac{1}{2}, \quad H = y(1-y^3), \tag{20}$$

$$n = -\tfrac{5}{4} \quad H = (1-y^3)(1-4y^3), \tag{21}$$

$$n = -\tfrac{7}{4}, \quad H = y(1-y^3)(1-\tfrac{5}{2}y^3), \tag{22}$$

$$n = -3, \quad H = (1-y^3)(1-11y^3+\tfrac{77}{5}y^6). \tag{23}$$

The function (20) was given by Riley (1958); the others are new. If (14) is integrated across the layer, from the surface to infinity, there results

$$Pr^{-1}H'(0) = -(4n+1)\int_0^{\infty} Hf'\,d\eta = -2(4n+1)\int_0^1 Hy\,dy. \tag{24}$$

Equation (24) relates the surface heat transfer to the integrated thermal energy content, convected downstream by the layer. Equations (20) and (22) give $H(0) = 0$, which implies that a wall jet hotter than ambient is passed over a surface maintained at ambient temperature. These flows heat the surface and $H'(0) > 0$. On the other hand, (21) and (23) satisfy $H(0) = 1$, but have no surface heat transfer; correspondingly the integral on the right of (24) vanishes. The physical interpretation of this is a cold jet blowing over a hot and insulated surface, with temperature excess and defect so arranged in the temperature profile that, when weighed with the local velocity, no net energy is transported downstream by convection.

The case $n = -2$ is of interest, because it provides the boundary between growth or decay in the downstream direction of the neglected direct driving term in (12). No simple closed-form solution was found for $n = -2$, but it was possible to establish that, with $H(0) = 1$, $H'(0) = -2{\cdot}08$. Thus the wall heats the flow. Equation (24) requires a negative energy integral; and it becomes less negative in the downstream direction. For constant heat flux along the surface,

† I am indebted to S. H. Maslen for pointing this out, and for obtaining (21)–(23).

$n = \frac{3}{4}$. In this case it was established that, with $H(0) = 1, H'(0) < 0$. The integral in (24) is now positive.

Clearly, there exist solutions of (13) and (14) for various n which satisfy the proper boundary conditions and are physically realistic. A full exploration of the possibilities is beyond the scope of this paper. Such solutions, together with the velocity field (15), are related to the original dimensional variables in (1)–(4) by

$$u = 4\left(g\frac{T_1-T_0}{T_0}\frac{L^2}{x}\right)^{\frac{1}{2}} f'(\eta),$$

$$w = \left(g\frac{T_1-T_0}{T_0}\frac{\nu^2L^2}{x^3}\right)^{\frac{1}{4}} (3\eta f'-f),$$

$$\Delta T = (T_1-T_0)\left(\frac{x}{L}\right)^n H(\eta).$$

$$\Delta p = \rho_0\left[g\frac{T_1-T_0}{T_0}\left(\frac{\nu}{L}\right)^{\frac{2}{3}} x\right]^{\frac{3}{4}}\left(\frac{x}{L}\right)^n G(\eta).$$

The similarity variable

$$\eta = z\left(g\frac{T_1-T_0}{T_0}\frac{L^2}{\nu^2x^3}\right)^{\frac{1}{4}}.$$

One should note that the G layer contains n only through temperature and pressure. In this flow, surface temperature variation does not affect the velocity field or the similarity variable, and therefore also not the boundary-layer thickness. This is in contrast to O or SRC layers. The maximum horizontal velocity occurs with $f' = 0.315$, and is

$$u_{\max_G} = 1{\cdot}26\left(g\frac{T_1-T_0}{T_0}\frac{L^2}{x}\right)^{\frac{1}{2}}. \tag{25}$$

A boundary-layer thickness δ is now defined on the basis that the velocity is 1 % of the maximum velocity. This occurs almost precisely at $\eta = 7$, therefore

$$\delta_G = 7\left(\frac{\nu^2x^3}{g(T_1-T_0)/T_0L^2}\right)^{\frac{1}{4}}. \tag{26}$$

Introduction of (26) into the above expression for pressure shows that

$$\Delta p_G = \frac{\rho_0}{7}\,\delta_G g\frac{T_1-T_0}{T_0}\left(\frac{x}{L}\right)^n G(\eta) = 0{\cdot}0898\rho_0 u^2_{\max_G}\frac{\delta_G}{L}\left(\frac{x}{L}\right)^{n+1} G(\eta). \tag{27}$$

Several points are worth noting. From (27), the pressure in the G layer is smaller, by a factor proportional to the boundary-layer thickness, than the dynamic pressure of the induced flow. It is the transverse pressure gradient that is of the order of the dynamic pressure. However, the pressure relative to the dynamic pressure varies in the downstream direction as $x^{n+\frac{7}{4}}$. If $n > -\frac{7}{4}$, the pressure increases, on this relative basis; and far enough downstream its neglect is no longer justified. This is a manifestation of the restriction in (12), that $\hat{G}r^{-\frac{1}{4}}\hat{x}^{\frac{7}{4}+n}$ be not large. If one extravagantly takes as the greatest permissible downstream distance for which the G layer remains valid that which makes this quantity unity, one has

$$\frac{x_{\max}}{L} = \left(g\,\frac{T_1 - T_0}{T_0}\,\frac{L^3}{\nu^2}\right)^{(7+4n)^{-1}}. \tag{28}$$

4. Considerations at the leading edge

The longitudinal velocity becomes infinite at the leading edge as $x^{-\frac{1}{2}}$. A physical interpretation is that the G layer has no thermal drive, direct or indirect, and therefore decelerates in the downstream direction. As a wall jet, it must get its drive from an externally imposed forcing at the leading edge.

Several kinds of drive come to mind. One, as in the original wall jet, involves impingement at the origin of a jet or sinking cold plume perpendicular to the surface. Also, an imposed horizontal jet can be visualized. Another drive could originate from an inclined portion of a heated surface located upstream, on which fluid directly driven by buoyancy (an O layer) proceeds towards the leading edge of the horizontal portion of the surface. This idea is explored below. The analysis shows how the length L, so far unspecified, can be related to physically meaningful parameters by a patching of G and O layers. This patching is performed simply by equating their maximum streamwise velocities and boundary-layer thicknesses. This procedure rests on the idea that the region of interaction between the layers is of the order of the boundary-layer thickness, and therefore short compared with the streamwise extent of the layers themselves. If this is so, then friction forces within it can be ignored, and the layers entering and leaving can be considered inviscid uniform streams, whose outer edge is a free streamline at constant velocity and pressure. The calculation is intended for a rough estimate of what one might anticipate from more rigorous matching calculations. An accuracy assessment is given in §7.

Suppose that at the upstream leading edge a semi-infinite horizontal surface is joined to an inclined straight surface of length l and angle α_0 to the horizontal. For simplicity, assume that this surface is isothermal and at temperature T_1. From the numerical results given by Ostrach (1964), one finds that the maximum longitudinal velocity and boundary-layer thickness (based on a value of 6·0 for the similarity variable used by Ostrach) of the O layer, in the present variables, are

$$u_{\max_0} = 0{\cdot}4\,(g(T_1 - T_0)/T_0\,\xi \sin\alpha_0)^{\frac{1}{2}}, \tag{29}$$

$$\delta_0 = 8{\cdot}5\left(\frac{\nu^2\xi}{g(T_1 - T_0)/T_0 \sin\alpha_0}\right)^{\frac{1}{4}}. \tag{30}$$

Here ξ is the distance along this inclined surface from its leading edge. At the junction with the horizontal surface, the velocity and boundary-layer thickness are given by (29) and (30) with $\xi = l$.

For the G layer, as pointed out by Glauert (1956, 1958), an arbitrary constant C can be introduced into (25) and (26) to shift the origin of the co-ordinate system to $x = -C$. Then in (25) and (26) one replaces x by $x + C$, and at the location

$x = 0$ one finds

$$u_{\max_G} = 1{\cdot}26\left(g\frac{T_1-T_0}{T_0}\frac{L^2}{C}\right)^{\frac{1}{2}}, \tag{31}$$

$$\delta_G = 7\left(\frac{\nu^2 C^3}{g(T_1-T_0)/T_0 L^2}\right)^{\frac{1}{4}}. \tag{32}$$

Equating (31) with (29), and (32) with (30), results in two equations relating C and L to the given length l. These are solved, to give

$$C = 0{\cdot}469l, \quad L = 0{\cdot}218l\sin^{\frac{1}{2}}\alpha_0.$$

One should properly equate suitably defined displacement thicknesses; but this refinement, which allows for differences in velocity profiles, has only a small effect on the results, and seems unjustified. Note also that allowance of several boundary-layer thicknesses for the size of the interaction region would supply only small corrections to these inviscid relations. Re-introducing them into (25) and (26), with the shifted co-ordinate system, results in

$$u_{\max_G} = 0{\cdot}4\left(g\frac{T_1-T_0}{T_0}l\sin\alpha_0\right)^{\frac{1}{2}}(2{\cdot}13x/l+1)^{-\frac{1}{2}}, \tag{33}$$

$$\delta_G = 8{\cdot}5\left(\frac{\nu^2 l}{g(T_1-T_0)/T_0\sin\alpha_0}\right)^{\frac{1}{4}}(2{\cdot}13x/l+1)^{\frac{3}{4}}. \tag{34}$$

For large x these expressions are, except for the numerical factors, just (25) and (26) with L replaced by $l\sin^{\frac{1}{2}}\alpha_0$. As one might have expected, the characteristic length L, which appears in the solution of the G layer, can be identified with a characteristic length of the imposed flow at the leading edge, which in this example is l. Although no complete matching between G and O layers has been attempted, the result is physically sufficiently plausible to give hope that the identification of L with $l\sin^{\frac{1}{2}}\alpha_0$ will survive rigorous calculation, fully accounting for departures from similarity in the neighbourhood of $x = 0$.

Unless $n \leqslant -\frac{7}{4}$, (33) and (34) will not be valid downstream of (28), which works out to be, for an isothermal surface,

$$x_{\max}/l = 0{\cdot}113\sin^{\frac{5}{7}}\alpha_0 Gr_l^{\frac{1}{7}} - 0{\cdot}469, \tag{35}$$

where

$$Gr_l = g\frac{T_1-T_0}{T_0}\frac{l^3}{\nu^2}. \tag{36}$$

This distance can be many multiples of l for sufficiently large Gr_l.

5. Connexion with indirectly driven flow and discussion

There have been many experiments reported in the literature on flow over a heated horizontal surface (as described in e.g. Pera & Gebhart 1973*a*), and some of these have been interpreted in terms of the SRC layer. None of the experiments is completely free of leading-edge effects. The results of § 4, for a G layer with upstream direct thermal driving, correspond to a horizontal flow over a heated surface which owes its existence entirely to leading-edge effects. It is therefore of interest to make a comparison between (33) and (34) and corresponding predictions for the SRC layer. The appropriate scaling from §2 results in

$$u = \left[\left(g\frac{T_1-T_0}{T_0}\right)^2 \nu x\right]^{\frac{1}{5}} \left(\frac{x}{L}\right)^{\frac{2}{5}n} F'(\eta),$$

$$\Delta p = \rho_0\left[\left(g\frac{T_1-T_0}{T_0}\right)^2 \nu x\right]^{\frac{2}{5}} \left(\frac{x}{L}\right)^{\frac{4}{5}n} G(\eta),$$

$$\eta = z\left(g\frac{T_1-T_0}{T_0}\nu^{-2}x^{-2}\right)^{\frac{1}{5}} \left(\frac{x}{L}\right)^{\frac{1}{5}n}.$$

Rotem & Claassen (1969) give isothermal wall results for various Pr. For $Pr = 1$ one gets, roughly, $F'_{\max} = 0{\cdot}6$ and $\eta = 8$ for the boundary-layer edge based on velocity. For the wall pressure, $G(0) = -1{\cdot}57$. Then, with $n = 0$, one has

$$u_{\max_{\rm SRC}} = 0{\cdot}6\,[(g(T_1-T_0)/T_0)^2\nu x]^{\frac{1}{5}} = 0{\cdot}212\,[g(T_1-T_0)/T_0\,\delta_{\rm SRC}]^{\frac{1}{2}}, \tag{37}$$

$$\delta_{\rm SRC} = 8\left(\frac{\nu^2x^2}{g(T_1-T_0)/T_0}\right)^{\frac{1}{5}}, \tag{38}$$

$$\Delta p_{{\rm wall}_{\rm SRC}} = -\,1{\cdot}57\rho_0\,[(g(T_1-T_0)/T_0)^2\,\nu x]^{\frac{2}{5}} = -\,12{\cdot}1\rho_0 u^2_{\max_{\rm SRC}}. \tag{39}$$

It is instructive to compare (37)–(39) with (25)–(27). The pressure in the SRC layer is proportional to the dynamic pressure, and this is obvious in view of the physical description of it given earlier. This is not because the pressure is larger in the SRC layer than in the G layer, but because the velocity is smaller (compare the second form on the right of (37) with (25)). The x dependence of velocity in either case is very different. Note also that L has disappeared from (37)–(39). This loss of a parameter has the consequence that it becomes impossible to patch directly velocity and boundary-layer thickness of an upstream O layer to the SRC layer at $x = 0$, by a method such as in §4. But this does become possible further downstream through an intervening G layer.

Comparing (37) with (25) indicates that G- and SRC-layer velocities can be equal when $\delta_{\rm SRC} = O(L^2/x)$. This condition turns out to agree with (28). That indicates a possible downstream matching between these layers. A patch, in the previous sense, is easily carried out, which equates the velocities from (37) and (33), and the boundary-layer thicknesses from (38) and (34). Two constants are introduced: one is a co-ordinate shift for the SRC layer, the other the co-ordinate of the matching point. Some algebra gives the latter to be about twice that of (35), or

$$x^*/l = 0{\cdot}287\sin^{\frac{5}{7}}\alpha_0 Gr_l^{\frac{1}{7}} - 0{\cdot}469. \tag{40}$$

There

$$u_{\max_G} = u_{\max_{\rm SRC}} = u^* = 0{\cdot}514\nu/l\sin^{\frac{1}{7}}\alpha_0 Gr_l^{\frac{3}{7}},$$

$$\delta_G = \delta_{\rm SRC} = \delta^* = 5{\cdot}89l\sin^{\frac{2}{7}}\alpha_0 Gr_l^{-\frac{1}{7}}.$$

Up to $x = x^*$, δ is given by (34); beyond, it is

$$\delta_{\rm SRC} = 8\left[\frac{\nu^2(x+0{\cdot}469l+0{\cdot}175l\sin^{\frac{5}{7}}\alpha_0 Gr_l^{\frac{1}{7}})^2}{g(T_1-T_0)/T_0}\right]^{\frac{1}{5}}.$$

For $x < x^*$, $\delta_G < \delta_{\rm SRC}$. Boundary-layer analysis is valid only if $\delta^*/(x^*+l)$ is small; this quantity is given by

$$\frac{\delta^*}{x^*+l} = \frac{20{\cdot}5}{\sin^{\frac{3}{7}}\alpha_0 Gr_l^{\frac{2}{7}}(1+1{\cdot}85\sin^{-\frac{5}{7}}\alpha_0 Gr_l^{-\frac{1}{7}})}. \tag{41}$$

l		Gr_l	x^*/l	$\frac{\delta^*}{x^*+l}$	u^* $(\mathrm{cm\,s^{-1}})$	$u_{\max_G}(0)$ $(\mathrm{cm\,s^{-1}})$
1 cm	Air	$4{\cdot}4\times10^{3}$	0·28	1·26	2·7	3·3
	Water	$0{\cdot}96\times10^{6}$	1·1	0·35	1·8	
10 cm	Air	$4{\cdot}4\times10^{6}$	1·5	0·24	4·3	10·4
	Water	$0{\cdot}9\times10^{9}$	3·9	0·057	3·6	
1 m	Air	$4{\cdot}4\times10^{9}$	4·9	0·038	10	33
	Water	$0{\cdot}96\times10^{12}$†	10·9	$8{\cdot}8\times10^{-3}$	6·5	
1 km	Air	$4{\cdot}4\times10^{18}$†	103	$1{\cdot}1\times10^{-4}$	72	$1{\cdot}04\times10^{3}$

† Significant for gravitationally stable conditions only.

TABLE 1

These expressions summarize results from patching an O layer to a G layer to an SRC layer. The G layer disappears if x^*/l from (40) vanishes, because then the O layer patches directly to the SRC layer. For this condition, (41) gives

$$\delta^*/(x^*+l) = 5{\cdot}06\sin\alpha_0.$$

This is clearly not small; therefore such a boundary-layer matching is impossible.

With Gr_l sufficiently large, a G layer can occupy an intermediate region $0 < x < x^*$. Table 1 gives some typical values of key quantities, assuming either air or water at 20 °C, $(T_1 - T_0)/T_0 = 10^{-1}$, and a straight, 45° inclined surface ahead of the leading edge of the horizontal surface.

The first entry in table 1 indicates that a typical laboratory experiment with a small, 1 cm characteristic thickness at the leading edge cannot be influenced by a G layer, since this is lost among general leading-edge effects. For this l, the layer in water is confined to $x \simeq l$, with the basis of the prediction only marginally valid, while in air the boundary-layer approximation breaks down. With $l = 1$ m, a G layer should be easily observable. The kilometre entry is included as of possible geophysical interest (discussed below).

Some remarks on stability, separation into vertical plumes, and turbulence are necessary, in view of some of the larger Gr_l given in table 1. Experiments by Tritton (1963), on a heated inclined plate, indicate that for a 45° inclination, laminar spells alternate with periods of fluctuating flow up to approximately $Gr = 10^9$ for gravitationally unstable flow above the surface, and $Gr = 3\times10^{10}$ for gravitationally stable flow below it. Beyond that, Tritton claimed some kind of turbulence becomes fully established. For this reason, conditions are flagged in table 1 for which the significance of laminar calculations is very questionable. They are nevertheless displayed for the following reasons.

Separation into vertical plumes cannot occur for flow above a cold surface. The present analysis is equally valid for horizontal flow driven by flow either up a hot incline or down a cold one. Consider then only the latter case for very large Gr_l. A local Reynolds number can be defined for the G layer by $u_{\max_G}\delta_G/\nu$. This increases with x, and attains its largest value at x^*. There one finds

$$Re_\delta^* = u^*\delta^*/\nu = 3{\cdot}03\sin^{\frac{3}{7}}\alpha_0\, Gr_l^{\frac{2}{7}}.$$

For $Gr_l = 10^{12}$, $\alpha_0 = \frac{1}{4}\pi$, this gives $Re_\delta^* = 7{\cdot}1 \times 10^3$. This is large, but not impossibly large for an unstratified laminar flow without a pressure gradient. Stable stratification will further stabilize the flow. Thus, even at this large Gr_l, the G layer above a cold surface could possibly stay laminar all the way up to x^*. But clearly this cannot be the case with $l = 1$ km. This example is given for its possible relevance to glacier winds. Tritton (1963) speculated that, even here, the flow may be intermittent, with periods of laminar flow during which the entire profile is locally well represented by a laminar solution. If this is so, then perhaps further speculation may be permitted, that the nature of the flow field over horizontal ground at the foot of a sloping glacier field is a wall jet. This seems intuitively obvious as a description of the decay of such a wind; and indeed the last row in table 1 shows a wall-jet region in effect extending to infinity. The laminar results of §4 may here find a geophysical application, which at the least could be a basis of comparison with future turbulent calculations.

6. Connexion with vertical buoyant plume

Consider now a two-dimensional G layer above a heated horizontal surface, with its origin at $x = -C_H$. Maximum velocity and boundary-layer thickness are then given by (25) and (26), with x replaced by $x + C_H$. Suppose that the layer separates to form a vertically rising buoyant plume, and let the horizontal distance to the plume centre-line be d. There is a known similarity solution for a two-dimensional buoyant plume driven by a line source of heat of strength Q (Brand & Lahey 1967; Gebhart *et al.* 1970; Fujii *et al.* 1973). A connexion between these flows can be made on the same basis as before, i.e. they can be patched together. When this is done, the nature of the patch turns out to be independent of Gr. Only the isothermal surface is considered. From the results of Fujii *et al.* (1973) and with a shift of the vertical co-ordinate z so that the plume origin is at $z = -C_V$, one obtains for the maximum velocity at the centre of the plume and the full plume width

$$w_{\max_F} = 0{\cdot}819 \left(\frac{gQ}{\rho_0 C_p T_0}\right)^{\frac{2}{5}} \left(\frac{z + C_V}{\nu}\right)^{\frac{1}{5}}, \tag{42}$$

$$\delta_F = 14 \left(\frac{gQ}{\rho_0 C_p T_0}\right)^{-\frac{1}{5}} [\nu^3 (z + C_V)^2]^{\frac{1}{5}}. \tag{43}$$

The quantity Q, which drives the plume, can be connected with the heat picked up by the G layer from the horizontal surface through

$$Q(x) = \rho_0 C_p \int_0^\infty u(T - T_0)\, dz.$$

This can be written as

$$\frac{gQ(x)}{\rho_0 C_p T_0} = 4 \left(g \frac{T_1 - T_0}{T_0}\right)^{\frac{1}{4}} (\nu L)^{\frac{1}{2}} (x + C_H)^{\frac{1}{4}} \int_0^\infty H f' d\eta.$$

The value of the integral is $\frac{1}{3}$ for an isothermal surface (see § 3); therefore the heat gained by the horizontal flow from its origin to the point of separation is

$$\frac{gQ}{\rho_0 C_p T_0} = \tfrac{4}{3}\left(g\frac{T_1 - T_0}{T_0}\right)^{\frac{5}{4}} (\nu L)^{\frac{1}{2}} (d + C_H)^{\frac{1}{4}}. \tag{44}$$

If one now uses (44) in (42) and (43), and equates these with (25) and (26), there results

$$d + C_H = 1{\cdot}89L, \quad C_V = 0{\cdot}727L.$$

These relations are independent of ν. They state that a G layer with characteristic length L, after proceeding above a heated surface for a distance approximately twice L, will have the same heat content, thickness, and characteristic velocity as a buoyant plume originating at a distance, roughly L, below the surface. The characteristic temperature differences of the two flows also are found to be compatible. The implication is not intended that this in some way demonstrates that the flow must separate there. Nevertheless, if it does, then a characteristic velocity at this point is given, by either (31) or (42), as

$$w^* = 0{\cdot}918\,[g(T_1 - T_0)/T_0 L]^{\frac{1}{2}}.$$

Anywhere above this point the maximum velocity and half-width of the plume are

$$w_{\max} = 0{\cdot}918\,[g(T_1 - T_0)/T_0 L]^{\frac{1}{2}} (1{\cdot}378z/L + 1)^{\frac{1}{5}},$$

$$\delta = 6{\cdot}24L\,Gr^{-\frac{1}{4}} (1{\cdot}378z/L + 1)^{\frac{2}{5}}.$$

These expressions combine the functional dependence on height of the classical plume solution with the dependence on gravity and viscosity of directly driven buoyant flow.

They also contain the so far unspecified quantity L, which represents an arbitrary constant with scales the strength of the G layer. There is an interesting case where this length can be related to the flow geometry. This is two-dimensional large Gr convection between infinite horizontal surfaces separated vertically by H. There have been attempts to construct such flows from a proper combination of horizontal and vertical boundary-layer-like flows (Robinson 1967, 1969, 1970; Wesseling 1969). Here the extent of the horizontal layer is also $O(H)$; and it may be that horizontal wall jets with L taken proportional to H, matched to ascending and descending plumes, play a role in determining the structure of this important example of buoyantly driven flow.

Finally, it should be noted that one finds that an O layer, for which the parameter L does not appear in either velocity or boundary-layer thickness, cannot because of this be patched to a plume by the present simple method.

7. Concluding remarks

The main point made in this paper is that the two-dimensional wall jet is relevant to two-dimensional natural convection flow over horizontal surfaces. Perhaps, in retrospect, this is obvious. The flow in its present context contains a scaling length L, which relates to conditions which drive the motion. This scaling factor is related to that of Glauert's original wall-jet solution. There, the quantity F occurs, which is a measure of the flow of exterior momentum, and a constant reference velocity $U \propto F\nu^{-2}$. One finds the correspondence to be

$$F = \tfrac{32}{5}\nu g \frac{T_1 - T_0}{T_0} L^2, \quad U = 256\left(g \frac{T_0 - T_0}{T_0} L\right)^{\frac{1}{2}} Gr^{\frac{1}{4}}.$$

Thus, by proper identification of F and U, the velocity field of Glauert's wall jet directly transforms into that discussed here, and perhaps these relations could have been guessed or established by dimensional analysis. In any case, the developments in §§ 2 and 3 establish this correspondence, and, further, the complementary temperature and pressure fields. This analysis is exact in the usual sense of boundary-layer theory.

On the other hand, the patching technique used in the remainder of the paper is not, and was not meant to be, rigorous. Although numbers are produced in these examples, they are to be interpreted only as approximate indications of what could be expected from refined matching calculations of the kind used by Jones (1973) in his matching of SRC and O layers on a slightly inclined surface. An accuracy estimate was obtained by carrying out the patching procedure for his problem and comparing this to his solution. This showed a wall shear 36 % too low at the location of the patch, where the maximum error occurs.

Generalization to axisymmetric flow would seem to present no difficulty in the case of a G layer driven by an upstream O layer. The corresponding solution for a diverging wall jet has been given by Glauert (1956), and directly driven flow on the outside of a vertical cylinder is also known (see again Ostrach 1964). However, geometrical generalization of the plume discussion in § 6 is another matter. The solution for an axisymmetric plume driven by a point source of heat is known. However, if this is fed by a horizontal inflow, it must be converging, and I know of no solution for a converging wall jet. Nor is it clear what could drive this. On the other hand, a descending cold plume could drive a divergent axisymmetric G layer. There is clearly room for further work.

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